

# Mathematical Modeling of Boson-Fermion Stars in the Generalized Scalar-Tensor Theories of Gravity\*

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## Abstract

A model of static boson-fermion star with spherical symmetry based on the scalar-tensor theory of gravity with massive dilaton field is investigated numerically.

Since the radius of star is *a priori* an unknown quantity, the corresponding boundary value problem (BVP) is treated as a nonlinear spectral problem with a free internal boundary. The Continuous Analogue of Newton Method (CANM) for solving this problem is applied.

Information about basic geometric functions and the functions describing the matter fields, which build the star is obtained. In a physical point of view the main result is that the structure and properties of the star in presence of massive dilaton field depend essentially both of its fermionic and bosonic components.

**Keywords.** boson-fermion star, scalar-tensor theory of gravity, massive dilaton field, two-parametric nonlinear spectral problem, continuous analog of Newton method, method of spline-collocation.

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## 1 Introduction

The most natural and promising generalizations of general relativity are the scalar-tensor theories of gravity [1] – [4]. In these theories gravity is mediated not only by a tensor field (the metric of space-time) but also by a scalar field (the dilaton). The scalar-tensor theories of gravity contain arbitrary functions of the scalar field that determine the gravitational “constant” as a dynamical variable and the strength of the coupling between the scalar field and matter. It should be stressed that specific scalar-tensor theories of gravity arise naturally as a low energy limit of the string theory [5] – [13], which is the most promising modern model of the unification of all fundamental physical interactions.

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If the string theory and its low energy limit are relevant to the real world, then the dilaton must be massive [14]. Unfortunately, our current understanding of how the dilaton acquires mass is primitive and it is tied to our lack of understanding of supersymmetry breaking. At present, we do not have a model of how the dilaton mass is generated in the string theory. Besides the mass term for dilaton field we may consider the general case of arbitrary dilaton potential, describing its nonlinear self-interaction.

From physical point of view, it is important to know how the dilaton mass and, in general, the dilaton potential influence the structure and stability of compact objects such as neutron stars, boson stars, and mixed fermion-boson stars.

It is known that the predictions of scalar-tensor theories of gravity with massless dilaton may differ drastically from these of general relativity. For example, the phenomenon of “spontaneous scalarization” was discovered recently [15], [16] as a non-perturbative strong field effect in a massive neutron star. The existence of this effect poses some important physical questions [17]. That is why it is natural to ask whether or not the “spontaneous scalarization” will occur when the dilaton is massive. In recent years, the boson stars in scalar-tensor theories of gravity with massless dilaton have been widely studied both analytically and numerically (see for example [18] - [25]). The study of boson stars in the case of massive dilaton is physically interesting and may be important for the understanding of their formation in the early universe.

The investigation of the compact objects in the generalized scalar-tensor theories of gravity helps us understand them better. On the other hand, the investigation of matter in extreme conditions like these in the neutron stars may demonstrate new phenomena and new features of specific scalar-tensor theories of gravity, originating from the low energy limit of the string theory. Thus, at first time we may be able to reach theoretical indications of physical manifestation of the string theory in the real world [26].

In the present paper we develop a direct numerical method for solving the equations of the general scalar-tensor theories of gravity including a dilaton potential term for the general case of mixed boson-fermion star.

The physical motivation for considering mixed boson-fermion stars is connected with the fact that many of the present-day existing stars are of primordial origin being formed from an original gas of fermions and bosons in the early universe. That is why it should be expected that they are a mixture of both fermions and bosons in different proportions. The study of such mixed objects is a new interesting problem, whose investigation was started in reference [27]. There exist different candidates for boson fields in stars such as Higgs field of Standard model, or axion field being a pseudoscalar partner of dilaton in the superstring theory. They are unavoidable part of modern physics, nevertheless up to now we have no experimental evidence for their existence. Taking into account that according to the modern understanding of the initial state of universe a significant amount of these fields must have been present during the Big Bang phase, one has to expect some part of these fields to be present in the stars of primordial origin. The study of new observable effects of boson fields in such mixed stars may give new ways for discovery of the existence of the above hypothetical fields, which at present are the most intriguing new objects in modern physics.

In the Einstein frame the field equations in the presence of fermion and boson matter are:

$$\begin{aligned}
G_i^j &= \kappa_* \left( \begin{matrix} B & F \\ T_i^j & T_i^j \end{matrix} \right) + 2\partial_i\varphi\partial^j\varphi & -\partial^l\varphi\partial_l\varphi\delta_i^j + \frac{1}{2}U(\varphi)\delta_i^j, \\
\nabla_i\nabla^i\varphi + \frac{1}{4}U'(\varphi) &= -\frac{\kappa_*}{2}\alpha\varphi \left( \begin{matrix} B & F \\ T & T \end{matrix} \right), \\
\nabla_i\nabla^i\Psi + 2\alpha(\varphi)\partial^l\varphi\partial_l\Psi &= -2A^2(\varphi)\frac{\partial\tilde{W}}{\partial\Psi^+}, \\
\nabla_i\nabla^i\Psi^+ + 2\alpha(\varphi)\partial^l\varphi\partial_l\Psi^+ &= -2A^2(\varphi)\frac{\partial\tilde{W}}{\partial\Psi^+},
\end{aligned} \tag{1}$$

where  $\nabla_i$  is the Levi-Civita connection with respect to the metric  $g_{ij}$ , ( $i = 0, \dots, 3$ ;  $j = 0, \dots, 3$ ). The constant  $\kappa_*$  is given by  $\kappa_* = 8\pi G_*$ , where  $G_*$  is the bare Newtonian gravitational constant. The physical gravitational “constant” is  $G_*A^2(\varphi)$ , where  $A(\varphi)$  is a function of the dilaton field  $\varphi$  depending on the concrete scalar-tensor theory of gravity. For example, in the framework of the Brans-Dicke model we have  $A(\varphi) = \exp(\frac{\varphi}{\sqrt{2\omega_{BD}+3}})$ , where  $\omega_{BD}$  is a parameter.

The dilaton potential  $U(\varphi)$  can be written in the form  $U(\varphi) = m_D^2 V(\varphi)$ , where  $m_D$  is the dilaton mass and  $V(\varphi)$  is a dimensionless model function of  $\varphi$ .

The complex scalar field  $\Psi$  describes the bosonic matter, while  $\Psi^+$  is its complex conjugated function. The quantity  $W(\Psi^+ \Psi)$  is the potential of boson field, which can be chosen in the following form:

$$\tilde{W}(\Psi^+ \Psi) = -\frac{m_B^2}{2} \Psi^+ \Psi - \frac{1}{4} \tilde{\Lambda} (\Psi^+ \Psi)^2,$$

where  $\tilde{\Lambda}$  is a parameter.

The scalar function  $\alpha(\varphi) = \frac{d}{d\varphi} [\ln A(\varphi)]$  determines the strength of the coupling between the dilaton field  $\varphi$  and matter.

The quantities  $\overset{B}{T}_i^j$  and  $\overset{F}{T}_i^j$  are correspondingly the trace of the energy-momentum tensor of the fermionic matter  $T_i^j$  and the bosonic matter  $T_i^j$ . We note that in the present article we consider the fermionic matter only in macroscopic approximation, *i.e.*, after averaging quantum fluctuations of the corresponding fermion fields. Thus, we actually consider standard classical relativistic matter.

The explicit forms of the mentioned tensors are correspondingly:

$$\begin{aligned} \overset{B}{T}_i^j = & \frac{1}{2} A^2(\varphi) (\partial_i \Psi^+ \partial^j \Psi + \partial_i \Psi \partial^j \Psi^+) \\ & - \frac{1}{2} A^2(\varphi) [\partial_l \Psi^+ \partial^l \Psi - 2A^2(\varphi) \tilde{W}(\Psi^+ \Psi)] \delta_i^j, \end{aligned} \quad (2)$$

$$\overset{F}{T}_i^j = (\varepsilon + p) u_i u^j - p \delta_i^j. \quad (3)$$

Here, the energy density and the pressure of the fermionic fluid in the Einstein frame are  $\varepsilon = A^4(\varphi) \tilde{\varepsilon}$  and  $p = A^4(\varphi) \tilde{p}$ , where  $\tilde{\varepsilon}$  and  $\tilde{p}$  are the physical energy density and pressure. Instead of giving the equation of state of the fermionic matter in the form  $\tilde{p} = \tilde{p}(\tilde{\varepsilon})$ , it is more convenient to write it in a parametric form:

$$\tilde{\varepsilon} = \tilde{\varepsilon}_0 g(\mu) \quad \tilde{p} = \tilde{\varepsilon}_0 f(\mu), \quad (4)$$

where  $\tilde{\varepsilon}_0$  is a properly chosen dimensional constant,  $\mu$  is the dimensionless Fermi momentum, and  $f(\mu)$  and  $g(\mu)$  are given functions (see below).

The physical four-velocity of the fermionic fluid is denoted by  $u_i$ .

The field equations together with the Bianchi identities lead to the local conservation law of the energy-momentum of matter:

$$\nabla_j \overset{F}{T}_i^j = \alpha(\varphi) \overset{F}{T} \partial_i \varphi. \quad (5)$$

From now on, we will take into consideration a static and spherically symmetric mixed boson-fermion star in asymptotic flat space-time. This means that the metric  $g_{ij}$  has the form:

$$ds^2 = e^{\nu(r)} dt^2 - e^{\lambda(r)} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (6)$$

where  $r, \theta, \phi$  are usual spherical coordinates.

The field configuration is static when the boson field  $\Psi$  satisfies the condition:

$$\Psi = \tilde{\sigma}(r) e^{i\omega t}.$$

Here,  $\omega$  is a real number and  $\tilde{\sigma}(r)$  is a real function.

Taking into account the above-stated assumption, the system of the field equations is reduced to a system of ordinary differential equations (ODEs). Before writing the system explicitly, we are going to introduce a rescaled (dimensionless) radial coordinate by  $r \rightarrow m_B r$ ,  $r \in [0, \infty)$ , where  $m_B$  is the mass of the bosons (a prime will denote the differentiation with respect to the dimensionless radial coordinate  $r$ ).

We also define the following dimensionless quantities by:

$$\Omega = \frac{\omega}{m_B}, \quad \sigma = \sqrt{\kappa_*} \tilde{\sigma}, \quad \Lambda = \frac{\tilde{\Lambda}}{\kappa_* m_B^2}, \quad \gamma = \frac{m_D}{m_B}.$$

The components of the energy-momentum tensors of the fermionic and bosonic matter, written by the dimensionless quantities, are correspondingly:

$$\overset{F}{T}_0^0 = bA^4(\varphi)g(\mu), \quad \overset{F}{T}_1^1 = \overset{F}{T}_2^2 = -bA^4(\varphi)f(\mu), \quad (7)$$

$$\overset{B}{T}_0^0 = \frac{1}{2}\Omega^2 A^2(\varphi)e^{-\nu}\sigma^2(r) + \frac{1}{2}A^2(\varphi)e^{-\lambda}\sigma'^2 - A^4(\varphi)W(\sigma^2), \quad (8)$$

$$\overset{B}{T}_1^1 = -\frac{1}{2}\Omega^2 A^2(\varphi)e^{-\nu}\sigma^2(r) - \frac{1}{2}A^2(\varphi)e^{-\lambda}\sigma'^2 - A^4(\varphi)W(\sigma^2), \quad (9)$$

$$\overset{B}{T}_2^2 = -\frac{1}{2}\Omega^2 A^2(\varphi)e^{-\nu}\sigma^2(r) + \frac{1}{2}A^2(\varphi)e^{-\lambda}\sigma'^2 - A^4(\varphi)W(\sigma^2). \quad (10)$$

The parameter  $b = \kappa_*\tilde{\varepsilon}_0/m_B^2$  describes the relation between the Compton length of dilaton and the usual radius of neutron star in general relativity.

It is necessary to note that two physically interesting borderline cases of pure bosonic and pure fermionic stars are formally contained in the above general system (1). For example, the model of pure bosonic stars can be obtained from (1) by letting the tensor  $\overset{F}{T}_i^j$  to be zero. While the pure fermionic stars correspond to the field  $\Psi \equiv 0$ . The case of pure bosonic stars in the scalar-tensor theories of gravity with a massive dilaton has already been discussed in our recent paper [28]. In the present paper we consider the mixed boson-fermion stars.

## 2 Formulation of the Problem

Under the physical assumptions we have made, the field equations (1) can be reduced to a system of ODEs. From mathematical point of view it is more convenient all ODEs to be of second order. That is why we first solve the Einstein equation  $G_1^1$  for  $e^\lambda$ :

$$e^\lambda = \frac{1 + r\nu' - r^2\varphi'^2 - \frac{1}{2}A^2(\varphi)r^2\sigma'^2}{1 - r^2 \left[ \overset{F}{T}_1^1 + \frac{1}{2}\gamma^2V(\varphi) - \frac{1}{2}\Omega^2 A^2(\varphi)e^{-\nu}\sigma^2 - A^4(\varphi)W(\sigma^2) \right]},$$

as a function of the quantities  $\nu(x)$ ,  $\nu'(x)$ ,  $\sigma(x)$ ,  $\sigma'(x)$ ,  $\varphi(x)$ ,  $\varphi'(x)$ , and the spectral parameter  $\Omega$ , and then substitute the above expression in the other Einstein equations. In this way, in terms of the dimensionless quantities, the system of the field equations (1) is reduced to the following system of ODEs:

$$\nu'' + \frac{\nu'}{r} = \left[ -\frac{\nu'}{r} + \left( \overset{F}{T}_0^0 - \overset{F}{T}_0^0 - 2\overset{F}{T}_2^2 + \overset{F}{T}_0^0 - \overset{B}{T}_1^1 - 2\overset{B}{T}_2^2 \right) \right. \\ \left. - \gamma^2V(\varphi) + \frac{\nu'r}{2}bigl(T_1 + \gamma^2V(\varphi)\bigr) \right] e^\lambda, \quad (11)$$

$$\varphi'' + \frac{\varphi'}{r} = \left[ -\frac{\varphi'}{r} + \frac{\alpha(\varphi)}{2} \left( \overset{F}{T} + \overset{B}{T} \right) \right. \\ \left. + \frac{1}{4}\gamma^2V'(\varphi) + \frac{\varphi'r}{2}(T_1 + \gamma^2V(\varphi)) \right] e^\lambda, \quad (12)$$

$$\sigma'' + \frac{\sigma'}{r} = -2\alpha(\varphi)\varphi'\sigma' + \left[ -\frac{\sigma'}{r} - 2A^2(\varphi)W'(\sigma^2)\sigma \right. \\ \left. - \Omega^2e^{-\nu}\sigma + \frac{\sigma'r}{2}(T_1 + \gamma^2V(\varphi)) \right] e^\lambda. \quad (13)$$

In the above equations, the potential of the bosonic matter  $W$  has the form:

$$W(\sigma^2) = -\frac{1}{2} \left( \sigma^2 + \frac{1}{2}\Lambda\sigma^4 \right),$$

and we suppose that  $W'(\sigma^2) \equiv \frac{dW}{d(\sigma^2)}$ . Similarly, we set  $V'(\varphi) \equiv \frac{dV}{d\varphi}$ .

The quantity  $T_1$  depends on the components of the energy-momentum tensors of the fermionic and bosonic matter (7)-(9):

$$T_1 = \overset{F}{T_0^0} + \overset{F}{T_1^1} + \overset{B}{T_0^0} + \overset{B}{T_1^1}.$$

The quantities  $\overset{B}{T}$  and  $\overset{F}{T}$  represent the traces of the these tensors, and are defined by the formulae:

$$\overset{B}{T} = -\Omega^2 A^2(\varphi) e^{-\nu} \sigma^2(r) + A^2(\varphi) e^{-\lambda} \sigma'^2 - 4A^4(\varphi) W(\sigma^2),$$

$$\overset{F}{T} = b A^4(\varphi) [g(\mu) - 3f(\mu)].$$

Correspondingly, the conservation law (5) can be expressed as:

$$\mu' = -\frac{g(\mu) + f(\mu)}{f'(\mu)} \left[ \frac{\nu'}{2} + \alpha(\varphi) \varphi' \right]. \quad (14)$$

The fermionic matter functions  $f(\mu)$  and  $g(\mu)$ , entered in the above relations, have the form:

$$f(\mu) = \frac{1}{8} \left[ (2\mu - 3) \sqrt{\mu + \mu^2} + 3 \ln \left( \sqrt{\mu} + \sqrt{1 + \mu} \right) \right], \quad (15)$$

$$g(\mu) = \frac{1}{8} \left[ (6\mu + 3) \sqrt{\mu + \mu^2} - 3 \ln \left( \sqrt{\mu} + \sqrt{1 + \mu} \right) \right]. \quad (16)$$

Let us now complete the problem by adding proper boundary conditions (BCs) to the system of differential equations (11)-(14).

The asymptotic flatness means that the function  $\nu(r) \rightarrow 0$  when  $r \rightarrow \infty$ . On the other hand, the nonsingularity condition at the center of the star requires the derivative  $\nu'(0) = 0$ . The same condition in relation to the dilaton field  $\varphi(r)$  implies that the derivative  $\varphi'(0) = 0$ . At the same time, the function  $\varphi(r)$  at the asymptotic infinity ( $r \rightarrow \infty$ ) must be  $\varphi_\infty = 0$  as it is required by the asymptotic flatness. The nonsingularity of the bosonic density  $\sigma(r)$  at the center of the star requires the derivative  $\sigma'(0) = 0$ . We need finite mass for the star, which implies  $\sigma(r) \rightarrow 0$  when  $r \rightarrow \infty$ . In addition, the central value  $\sigma_c = \sigma(0)$  must be given. Concerning the fermionic fluid, we have to give the central density  $\tilde{\varepsilon}_c = \tilde{\varepsilon}(0)$  or, equivalently, the central value  $\mu_c = \mu(0)$ .

It should be noted that for the physically relevant equation of state of the fermionic matter there must be a point  $r = R_s < \infty$ , where the pressure of the fermionic matter vanishes, *i.e.*,  $R_s$  is the radius of the fermionic part of the star.

As a conclusion, from the above-mentioned physical assumptions, we can formulate the following linear boundary conditions (BCs) for the quantities under consideration:

$$\nu'(0) = 0, \quad \nu(\infty) = 0; \quad (17)$$

$$\varphi'(0) = 0, \quad \varphi(\infty) = 0; \quad (18)$$

$$\sigma'(0) = 0, \quad \sigma(\infty) = 0; \quad (19)$$

$$\mu(0) = \mu_c. \quad (20)$$

Here, we denote  $(\cdot)(\infty) \stackrel{\text{def}}{=} \lim_{r \rightarrow \infty} (\cdot)(r)$ .

Apart from the unknown functions  $\nu(r)$ ,  $\sigma(r)$ ,  $\varphi(r)$ , and  $\mu(r)$ , the equations (11)-(14) also include two unknown real parameters,  $R_s > 0$  and  $\Omega$ . However, the seven BCs (17)-(20) are insufficient for their computation. In order to determine these parameters, we have to use additional conditions. In other words, the problem may be considered as a nonlinear eigenvalue problem, where  $R_s$  and  $\Omega$  are considered as “eigenvalues”. For this purpose, further on we use two physically-clear additional conditions.

The first one, given by the relation:

$$\sigma(0) = \sigma_c \quad (21)$$

determines the density  $\sigma_c \geq 0$  of the bosonic matter in the star's center. The second one:

$$\mu(R_s) = 0, \quad 0 < R_s < \infty \quad (22)$$

describes the condition that the density of the fermionic matter must vanish at the radius of the star.

Finally, we note that all the functions  $\nu(r)$ ,  $\sigma(r)$ , and  $\varphi(r)$  are defined in the whole real half-line  $r \in [0, \infty)$ . It is easy to see that these functions are smooth in this interval including the point  $x = 1$ . Whereas, the fermionic density  $\mu(r)$  is defined and smooth only inside the star, *i.e.*,  $r \in [0, R_s]$ .

### 3 Method of Solution

For solving the above formulated nonlinear eigenvalue problem the Continuous Analogue of Newton Method (CANM) (see [29] - [33], and comprehensive surveys [34], [35]) is applied. For convenience, a brief description of CANM can be found in the attached Appendix.

The presence of the *a priori* unknown quantity  $R_s$ , however, is an obstacle for the direct use of CANM - the problem is the unknown internal boundary  $R_s$ . In order to overcome this obstacle, we introduce a new scaled coordinate  $x = r/R_s$ . As a result, the physical domain  $r \in [0, \infty)$  renders to the domain  $x \in [0, \infty)$ , and the star's radius  $r = R_s$  maps into the fixed point  $x = 1$ . Then the BC (22) for  $\mu(x)$  becomes

$$\mu(1) = 0. \quad (23)$$

Let  $x_1$  and  $x_2$  be two arbitrary points in the internal domain  $[0, 1]$ . We note that for the arbitrary functions  $f(\mu)$ ,  $g(\mu)$ , and  $\alpha(\varphi)$  the equation (14) has a first integral, which can be presented as:

$$\int_{\mu_1}^{\mu_2} \frac{f'(\mu)}{f(\mu) + g(\mu)} d\mu + \frac{1}{2} (\nu_2 - \nu_1) + \ln \frac{A(\varphi_2)}{A(\varphi_1)} = 0,$$

where  $\nu_1, \nu_2, \varphi_1, \varphi_2, \mu_1, \mu_2$  stand for the functions  $\nu(x)$ ,  $\varphi(x)$ ,  $\mu(x)$  at the points  $x_1$  and  $x_2$ , respectively. Thus, for the model of the fermionic matter described by the conditions (15), (16) we simply get the following algebraic equation:

$$\ln \left[ \frac{(1 + \mu_2)A^2(\varphi_2)}{(1 + \mu_1)A^2(\varphi_1)} \right] + \nu_2 - \nu_1 = 0. \quad (24)$$

For convenience, we introduce the vector  $\mathbf{y}(x) = \{\nu(x), \varphi(x), \sigma(x)\}$ . Then the first three equations (11) - (13) of the problem and the corresponding BCs (17) - (19) can be rewritten as follows:

$$-x\mathbf{y}'' - \mathbf{y}' + \mathbf{F} = 0, \quad (25)$$

$$\mathbf{y}'(0) = 0, \quad \mathbf{y}(\infty) = 0, \quad (26)$$

where  $\mathbf{F} = \mathbf{F}(x, \mathbf{y}, \mathbf{y}', \mu, R_s, \Omega)$  is 3D vector consisting of the right-hand sides (RHSs) of the equations (11)-(13) multiplied by  $R_s^2 x$ . The differentiation with respect to the new independent variable  $x$  is denoted by  $(\cdot)'$ . In the linear case, the advantages of such representation of the radial operator are discussed in [36].

Following CANM, we introduce a “time-like” parameter  $t \in [0, \infty)$  and assume the unknown quantities depend on  $t$  as well:  $\mathbf{y} = \mathbf{y}(x, t)$ ,  $R_s = R_s(t)$ ,  $\Omega = \Omega(t)$ . Let us suppose that the function  $\mu = \mu(x)$  is known (see below). Then the CANM equations [35] corresponding to (25) and (26) become:

$$\begin{aligned} -x\mathbf{z}'' + \left( \frac{\partial \mathbf{F}}{\partial \mathbf{y}'} - \mathbf{E} \right) \mathbf{z}' + \frac{\partial \mathbf{F}}{\partial \mathbf{y}} \mathbf{z} + \left( \frac{2}{R_s} \mathbf{F} + \frac{\partial \mathbf{F}}{\partial R_s} \right) \rho + \frac{\partial \mathbf{F}}{\partial \Omega} \omega \\ = x\mathbf{y}'' + \mathbf{y}' - \mathbf{F}. \end{aligned} \quad (27)$$

$$\mathbf{z}'(0) = -\mathbf{y}'(0), \quad \mathbf{z}(\infty) = -\mathbf{y}(\infty), \quad (28)$$

where  $\mathbf{E}$  is an identity  $3 \times 3$  matrix and

$$\dot{\mathbf{y}} = \mathbf{z}, \quad \dot{R}_s = \rho, \quad \dot{\Omega} = \omega. \quad (29)$$

The respective Frechét derivatives at the point  $(\mathbf{y}, R_s, \Omega)$  are  $\partial \mathbf{F} / \partial (\cdot)$  and the dot in (29) and below denotes the differentiation with respect to “time”  $t$ .

The solution  $\mathbf{z}(x)$  of the above equation is sought as a linear function towards the derivatives  $\rho$  and  $\omega$

$$\mathbf{z} = \mathbf{u} + \rho \mathbf{v} + \omega \mathbf{w}, \quad (30)$$

where  $\mathbf{u}(x)$ ,  $\mathbf{v}(x)$ , and  $\mathbf{w}(x)$  are supposed to be new unknown 3D vector-functions of  $x$ . Substituting for them in equation (27), we obtain the following three vector ODEs of second order with respect to these quantities:

$$-x\mathbf{u}'' - \mathbf{u}' + \frac{\partial \mathbf{F}}{\partial \mathbf{y}}\mathbf{u} + \frac{\partial \mathbf{F}}{\partial \mathbf{y}'}\mathbf{u}' = x\mathbf{y}'' + \mathbf{y}' - \mathbf{F} \quad (31)$$

$$-x\mathbf{v}'' - \mathbf{v}' + \frac{\partial \mathbf{F}}{\partial \mathbf{y}}\mathbf{v} + \frac{\partial \mathbf{F}}{\partial \mathbf{y}'}\mathbf{v}' = -\left(\frac{2}{R_s}\mathbf{F} + \frac{\partial \mathbf{F}}{\partial R_s}\right) \quad (32)$$

$$-x\mathbf{w}'' - \mathbf{w}' + \frac{\partial \mathbf{F}}{\partial \mathbf{y}}\mathbf{w} + \frac{\partial \mathbf{F}}{\partial \mathbf{y}'}\mathbf{w}' = -\frac{\partial \mathbf{F}}{\partial \Omega}. \quad (33)$$

The above three equations are coupled with the following six BCs:

$$\mathbf{u}'(0) = -\mathbf{y}'(0), \quad \mathbf{u}(\infty) = -\mathbf{y}(\infty); \quad (34)$$

$$\mathbf{v}'(0) = 0, \quad \mathbf{v}(\infty) = 0; \quad (35)$$

$$\mathbf{w}'(0) = 0, \quad \mathbf{w}(\infty) = 0, \quad (36)$$

which are obtained from BCs (28), substituting for them with decomposition (30) also. Let us emphasize that the above equations (31)-(36) have equivalent structures of the left-hand sides, which essentially facilitates their numerical treatment.

In order to calculate the derivatives  $\rho$  and  $\omega$ , we apply CANM for the first additional BC (21). This gives:

$$\dot{\sigma}(0) = \sigma_c - \sigma(0).$$

One more condition is required. Unfortunately, the second additional condition (23) is not convenient for this purpose because knowledge about decomposition (30) concerning the function  $\mu(x)$  is not available. We avoid this difficulty using the integral (24) for  $x_1 \equiv 0$  and  $x_2 \equiv 1$ . Taking into account conditions (20) and (23), we obtain an algebraic equation with respect to the quantities  $\nu(0)$ ,  $\nu(1)$ ,  $\varphi(0)$ ,  $\varphi(1)$ . After applying CANM to this equation, we get

$$\begin{aligned} \dot{\nu}(1) - \dot{\nu}(0) + 2 \frac{A'[\varphi(1)]}{A[\varphi(1)]} \dot{\varphi}(1) - 2 \frac{A'[\varphi(0)]}{A[\varphi(0)]} \dot{\varphi}(0) = \\ \ln(1 + \mu_c) - [\nu(1) - \nu(0)] - 2 \ln \frac{A[\varphi(1)]}{A[\varphi(0)]} = 0, \end{aligned}$$

where the abbreviation  $A'$  denotes the derivative of the function  $A$  with respect to the argument  $\varphi$ .

Let us now eliminate all the derivatives in relation to “time”  $t$  by means of decomposition (30). As a result, we receive the following linear system of algebraic equations:

$$\begin{aligned} a_1\rho + b_1\omega &= c_1 \\ a_2\rho + b_2\omega &= c_2 \end{aligned} \quad (37)$$

with respect to the unknown derivatives  $\rho$  and  $\omega$ . The coefficients in formulae (37) are given by:

$$\begin{aligned} a_1 &= v_1(1) - v_1(0) + 2 \frac{A'[\varphi(1)]}{A[\varphi(1)]} v_2(1) - 2 \frac{A'[\varphi(0)]}{A[\varphi(0)]} v_2(0) \\ b_1 &= w_1(1) - w_1(0) + 2 \frac{A'[\varphi(1)]}{A[\varphi(1)]} w_2(1) - 2 \frac{A'[\varphi(0)]}{A[\varphi(0)]} w_2(0) \\ c_1 &= \ln[1 + \mu_c] - [\nu(1) - \nu(0)] - 2 \frac{A'[\varphi(1)]}{A[\varphi(1)]} u_2(1) + 2 \frac{A'[\varphi(0)]}{A[\varphi(0)]} u_2(0) \\ &\quad - 2 \ln \frac{A[\varphi(1)]}{A[\varphi(0)]} - u_1(1) + u_1(0) \\ a_2 &= v_3(0), \quad b_2 = w_3(0), \quad c_2 = \sigma_c - \sigma(0) - u_3(0). \end{aligned}$$

Obviously, the explicit form of the coefficients in system (37) depends on the concrete choice of functions  $f(\mu)$  and  $g(\mu)$ .

## 4 General Sequence of the Algorithm

We discretize the continuous “time-like” parameter  $t \in [0, \infty)$  in the following way:  $t_{k+1} = t_k + \tau_k$ ,  $t_0 = 0$ , where  $k = 0, 1, 2, \dots$  denotes the number of iterations, and the “time” step  $\tau_k$  is generally assumed as a variable quantity. Next, we use the Euler difference scheme [34] to approximate the “time” derivatives in equations (29). Then we can write:

$$\begin{aligned} \mathbf{y}_{k+1}(x) &= \mathbf{y}_k(x) + \tau_k [\mathbf{u}_k(x) + \rho_k \mathbf{v}_k(x) + \omega_k \mathbf{w}_k(x)], \\ R_{s,k+1} &= R_{s,k} + \tau_k \rho_k, \\ \Omega_{k+1} &= \Omega_k + \tau_k \omega_k. \end{aligned} \quad (38)$$

Let us suppose that the functions  $\nu_k(x)$ ,  $\varphi_k(x)$ ,  $\sigma_k(x)$ ,  $\mu_k(x)$  and the parameters  $R_{s,k}$ ,  $\Omega_k$  are given. We solve the linear BVP (31)-(33) and, thus, we compute the functions  $\mathbf{u}_k(x)$ ,  $\mathbf{v}_k(x)$ ,  $\mathbf{w}_k(x)$ . Next, to obtain the derivatives  $\rho_k$  and  $\omega_k$  we solve system (37). After that, using decomposition (38) for a selected  $\tau_k$ , we calculate the functions  $\nu_{k+1}(x)$ ,  $\varphi_{k+1}(x)$ ,  $\sigma_{k+1}(x)$ , the radius of the star  $R_{s,k+1}$ , and the quantity  $\Omega_{k+1}$  as well at the new stage  $k + 1$ . In the end, we calculate the function  $\mu_{k+1}(x)$  at the new stage, according to the recurrent formula, which can be obtained immediately from the first integral (24).

For every iteration  $k$  an optimal time step  $\tau_{opt}$  is determined in accordance to the Kalitkin&Ermakov formula [37], [33]:

$$\tau_{opt} = \frac{\delta(0)}{\delta(0) + \delta(1)}, \quad (39)$$

where the residual  $\delta(\tau)$  is represented as follows:

$$\delta(\tau_k) = \max [\delta_f, (R_{s,k} + \tau_k \rho_k)^2, (\Omega_k + \tau_k \omega_k)^2]$$

and  $\delta_f$  is the Euclidean residual of RHS of the equation (31). Formula (39) provides approximately the minimal value of the residual for the current solution, given by (38).

The criterion for termination of the iterations is  $\delta(\tau_{opt}) < \varepsilon$ , where  $\varepsilon \sim 10^{-8} \div 10^{-12}$ . Then, for the sought solutions we set  $\nu(x) \equiv \nu_{k+1}(x)$ ,  $\varphi(x) \equiv \varphi_{k+1}(x)$ ,  $\sigma(x) \equiv \sigma_{k+1}(x)$ ,  $R_s \equiv R_{s,k+1}$ ,  $\Omega \equiv \Omega_{k+1}$ .

The use of the standard programs available, for example, via the Internet [40], to solve numerically the linear BVPs (31)-(36) is unhandy for many reasons. Because of that, the spline-collocation scheme is employed in our case.

We introduce a nonuniform grid

$$\Delta: x_{i+1} = x_i + h_i, \quad i = 0, 1, \dots, N_s, N_{s+1}, \dots, N-1, \quad x_0 = 0, \quad x_N = X_\infty,$$

on the interval  $x \in [0, X_\infty]$ , condensing to the points  $x = 0$  and  $x = 1$ . Here,  $X_\infty$  is the “actual infinity”,  $N_s$  is the number of the node  $x = 1$ ,  $N$  is the full number of the subintervals, and  $h_i$  is the grid step. We will seek approximate solutions of the above linear BVPs as a cubic spline on the grid  $\Delta$ . Namely, for  $x \in [x_i, x_{i+1}]$ ,  $i = 0, \dots, N-1$  we set

$$\mathbf{U}(x) = \psi_1(\theta) \mathbf{U}_i + \psi_2(\theta) \mathbf{M}_i + \psi_3(\theta) \mathbf{U}_{i+1} + \psi_4(\theta) \mathbf{M}_{i+1}. \quad (40)$$

In the above formula the relative coordinate  $\theta = (x - x_i)/h_i$  and the known functions  $\psi_l(\theta)$ ,  $l = 1, \dots, 4$ , are the coefficients of the spline. For simplicity in the last formula, we introduced the  $3 \times 3$  matrices  $\mathbf{U}$  and  $\mathbf{M}$ , consisting of the coordinates of the vectors  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  from (30) and their first moments at the spline nodes  $x_i$ ,  $i = 0, \dots, N$ . According to the collocation method [38], in every subinterval  $[x_i, x_{i+1}]$ ,  $i = 0, \dots, N-1$ , the system (31) - (33) is satisfied at the corresponding Gaussian points  $\theta_1 = 1/2 - \sqrt{3}/6$  and  $\theta_2 = 1/2 + \sqrt{3}/6$ . This kind of discretization yields an algebraic system with respect to the functions and their moments at the spline nodes. The corresponding matrix has an almost block-diagonal structure (see [38]). Therefore, at the  $i$ -th block ( $i = 1, \dots, N-1$ ) the collocation equations have the form:

$$\begin{pmatrix} \|a_{kn}^1\| & \|b_{kn}^1\| & \|c_{kn}^1\| & \|d_{kn}^1\| \\ \|a_{kn}^2\| & \|b_{kn}^2\| & \|c_{kn}^2\| & \|d_{kn}^1\| \end{pmatrix} \begin{pmatrix} \mathbf{U}_i \\ \mathbf{M}_i \\ \mathbf{U}_{i+1} \\ \mathbf{M}_{i+1} \end{pmatrix} = \begin{pmatrix} \mathbf{e}_i^1 \\ \mathbf{e}_i^2 \end{pmatrix},$$

where  $\mathbf{e}_i$  is the vector of RHSs of the equations (31)-(33) at the collocation nodes, while the superscript corresponds to the number of these nodes. Here:

$$\begin{aligned}
{}_{kn}^j &= - \left( \frac{\xi_{ij}}{h_i^2} \ddot{\psi}_{1j} + \frac{1}{h_i} \dot{\psi}_{1j} \right) \delta_{kn} + \left( \frac{\partial F_k}{\partial y'_n} \right)_{ij} \frac{1}{h_i} \dot{\psi}_{1j} + \left( \frac{\partial F_k}{\partial y_n} \right)_{ij} \psi_{1j}, \\
b_{kn}^j &= - \left( \frac{\xi_{ij}}{h_i} \ddot{\psi}_{2j} + \dot{\psi}_{2j} \right) \delta_{kn} + \left( \frac{\partial F_k}{\partial y'_n} \right)_{ij} \dot{\psi}_{2j} + \left( \frac{\partial F_k}{\partial y_n} \right)_{ij} h_i \psi_{2j}, \\
{}_{kn}^j &= - \left( \frac{\xi_{ij}}{h_i^2} \ddot{\psi}_{3j} + \frac{1}{h_i} \dot{\psi}_{3j} \right) \delta_{kn} + \left( \frac{\partial F_k}{\partial y'_n} \right)_{ij} \frac{1}{h_i} \dot{\psi}_{3j} + \left( \frac{\partial F_k}{\partial y_n} \right)_{ij} \psi_{3j}, \\
d_{kn}^j &= - \left( \frac{\xi_{ij}}{h_i} \ddot{\psi}_{4j} + \dot{\psi}_{4j} \right) \delta_{kn} + \left( \frac{\partial F_k}{\partial y'_n} \right)_{ij} \dot{\psi}_{4j} + \left( \frac{\partial F_k}{\partial y_n} \right)_{ij} h_i \psi_{4j}, \\
\text{for } & k = 1, 2, 3, \quad n = 1, 2, 3, \quad j = 1, 2,
\end{aligned}$$

and the quantities  $\xi_{ij} = x_i + \theta_j h_i$  are the absolute coordinates of the collocation points. The derivatives of the spline coefficients  $\psi_l$  with respect to the relative coordinate  $\theta$  are dotted.

The dimensions of the first and the last blocks in the global matrix are greater, since we add two matrix rows corresponding, respectively, to the left and right BCs.

Formula (40) is also used for the approximation of the RHSs of system (31) - (33) in the collocation points.

The spline-difference schemes of this kind have a high order of approximation  $\mathcal{O}(\bar{h}^4)$ , where  $\bar{h} = \max\{h_i\}$ ,  $i = 0, \dots, N$ .

It is clear that for solving all the three algebraic systems, corresponding to the linear BVPs (31) - (36) at every iteration, only one *LU*-decomposition is necessary.

Depending on the initial values of the government physical parameters, the number of iterations varies approximately in the range 4  $\div$  16. If we vary some solution as a function of one of the parameters  $\mu_c$ ,  $\sigma_c$ ,  $\gamma$ ,  $\Lambda$ , or  $b$ , then we use the previous solution as an initial approximation for computing the next one.

## 5 Results and Discussion

In order to be specific in the present article, we focus our attention on a concrete scalar-tensor gravity model, characterized by the functions

$$A(\varphi) = \exp\left(\frac{\varphi}{\sqrt{3}}\right) \quad \text{and} \quad V(\varphi) = (1 - [A(\varphi)]^{-1})^2.$$

For more details concerning this gravitational model, we refer the reader to the recent paper [39] and the references therein.

The order of approximation of the used spline-difference scheme is verified by the Runge rule.

The Runge rule is presented by the formula:

$$\frac{y_h - y_{\frac{h}{2}}}{y_{\frac{h}{2}} - y_{\frac{h}{4}}} = 2^p,$$

where  $p$  is Runge's number and  $y_h$ ,  $y_{\frac{h}{2}}$ ,  $y_{\frac{h}{4}}$  are the values of the grid function  $y$  at the given node, computed on meshes with steps  $h$ ,  $h/2$ , and  $h/4$ . In our case  $p$  must be approximately equal to 4.

In Table 1 the values of the sought grid functions at the point  $x = 1$ , the corresponding radius of star  $R_s$ , and the quantity  $\Omega$  for  $\sigma_c = 0.8$ ,  $\mu_c = 1$ ,  $\Lambda = 0.01$ ,  $\gamma = 1$ ,  $b = 1$ , and  $X_\infty = 128$ , are shown.

Therefore, it is obvious that the Runge relationship is satisfied both for the functions and the eigenvalues  $R_s$  and  $\Omega$ .

The correctness of the spline-difference scheme is verified through appropriate numerical experiments consisting of both grid doubling and doubling of the “actual infinity”. For this purpose, uniform meshes are used with numbers of the spline nodes  $N = 256, 512, 1024, 2048$ , respectively. It turns out that the relative error between the values of the functions  $\nu(x)$ ,  $\varphi(x)$ , and  $\sigma(x)$ , varies in the range 0.1% – 1% when the mesh is “coarse” ( $N = 256, 512$ ), and in the range 0.003% – 0.02% when the mesh is “fine” ( $N = 1024, 2048$ ). Similar experiments are carried out with the “actual infinity”

Table 1: Data for checking the Runge rule.

$h$	$\nu(1)$	$\varphi(1)$	$\sigma(1)$	$R_s$	$\Omega$
$\frac{1}{16}$	-1.0059230404	-0.0471137759	0.4777335163	1.1609111685	0.8006662485
$\frac{1}{32}$	-1.0059334054	-0.0471120738	0.4777483180	1.1608888836	0.8006671950
$\frac{1}{64}$	-1.0059342032	-0.0471119781	0.4777490917	1.1608875328	0.8006672467
$p$	3.61	4.22	4.37	4.06	4.28

$X_\infty = 64, 128, 256$ . It is interesting to note that the relative error between the set functions  $\varphi(x)$  and  $\sigma(x)$  is very small (less than  $10^{-4}\%$ ), while the function  $\nu(x)$  is more sensitive with respect to the choice of the quantity  $X_\infty$ . This fact is fully explainable if we take into account that the function  $\nu(x)$  decreases slowly at the infinity compared to the other functions. (Theoretically  $\nu(x) \sim -\frac{M}{R_s x}$  when  $x \rightarrow \infty$ . Here, the quantity  $M$  is the total star mass.) The computed values of the derivative  $\nu'(X_\infty)$  as a function of the “actual infinity”  $X_\infty$  are presented in Table 2. It is easy to see the relationship  $\nu'(X_\infty) = \frac{C}{X_\infty^2}$ , where the constant  $C > 0$  depends on the concrete solution (for the above solution  $C \approx 1.133$ ).

 Table 2: Asymptotic behaviour of the derivative  $\nu'$  at the “actual infinity”  $X_\infty$ .

$X_\infty$	32	64	128	256	512
$\nu'(X_\infty)$	$1.07246 \times 10^{-3}$	$2.63721 \times 10^{-4}$	$6.53945 \times 10^{-5}$	$1.62825 \times 10^{-5}$	$4.06241 \times 10^{-6}$

All government parameters are varied in wide physically-admissible ranges. As initial distributions of the functions  $\nu(x)$ ,  $\varphi(x)$ ,  $\sigma(x)$  and  $\mu(x)$  both analytic and numerical approximations are used.

Results concerning a family of solutions will be considered below. They are obtained for the following fixed values of the parameters:  $\mu_c = 0.5$ ,  $\Lambda = 10$ ,  $\gamma = 10$ ,  $b = 1$ , and the “actual infinity”  $X_\infty = 128$ , when the parameter  $\sigma_c$  runs the interval  $[0.1, 0.9]$ .

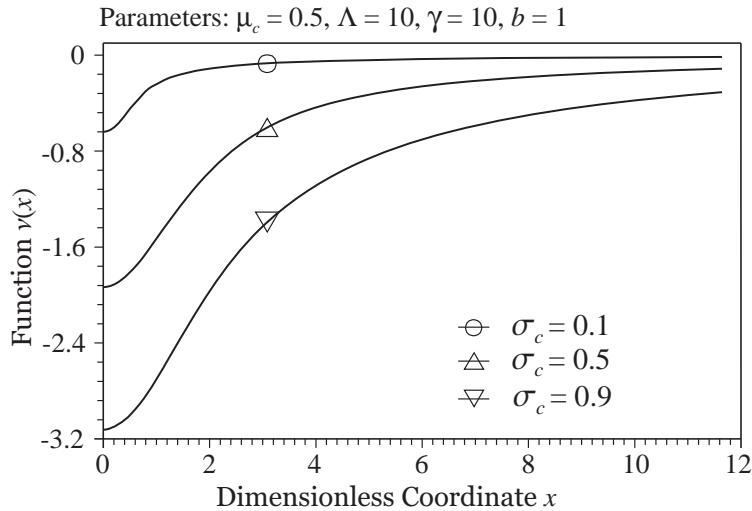


Figure 1: The function  $\nu(x)$  in dependence on the parameter  $\sigma_c$ : “ $\circ$ ” -  $\sigma_c = 0.1$ ; “ $\triangle$ ” -  $\sigma_c = 0.5$ ; “ $\nabla$ ” -  $\sigma_c = 0.9$ .

Figure 1 presents the dependence of the function  $\nu(x)$  on the dimensionless coordinate  $x$  for three different values of the central bosonic density  $\sigma_c$ . It is seen that when  $\sigma_c$  increases, the absolute value of  $\nu(x)$  as a whole decreases and at great distances (from 3 star radii when  $\sigma_c = 0.1$  until 45 star radii in the case  $\sigma_c = 0.9$ ) from the star’s center approaches asymptotically zero. The qualitative behaviour of the three curves, however, remains the same. Such a behaviour is natural and should be expected if the differential equation (11) for  $\nu(r)$  is taken into account. From physical point of view, this behaviour is natural also because the function  $\exp(\frac{\nu(x)}{2})$  is related to the gravitational potential.

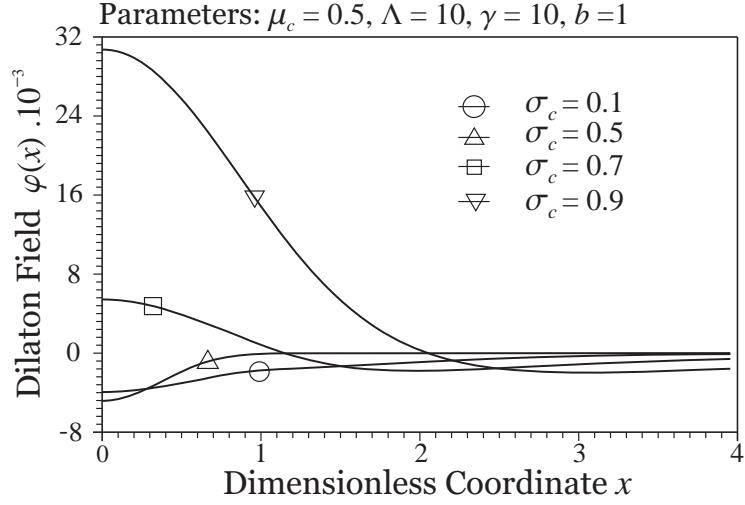


Figure 2: The potential of dilaton  $\varphi(x)$  as function of the parameter  $\sigma_c$ : “ $\circ$ ” -  $\sigma_c = 0.1$ ; “ $\triangle$ ” -  $\sigma_c = 0.5$ ; “ $\square$ ” -  $\sigma_c = 0.7$ ; “ $\nabla$ ” -  $\sigma_c = 0.9$ .

Figure 2 presents the dependence of the dilaton field  $\varphi(x)$  on the dimensionless coordinate  $x$  for four different values of  $\sigma_c$ . The qualitative behaviour of the field  $\varphi(x)$  as a function of  $\sigma_c$  is the following. For small values when  $\sigma_c$  increases, the dilaton field around the center of the star decreases. Then, after some critical value  $\sigma_c^*$  the behaviour of  $\varphi(x)$  is changed and  $\varphi(x)$  around the center of the star begins to increase with the increase of  $\sigma_c$ . The cause of the described behaviour is the presence of the term  $\overset{B}{T}$  on the RHS of equation (12). For sufficiently small values of the density  $\sigma_c$  the term  $\overset{B}{T}$  is negative and has a dominant contribution with respect to the term  $\overset{F}{T}$ . For the sufficiently large central value  $\sigma_c$  ( $\sigma_c \geq \sigma_c^*$ ), the term  $\overset{B}{T}$  changes its sign and amplifies the contribution of  $\overset{F}{T}$ , leading to the increase of the function  $\varphi(x)$ .

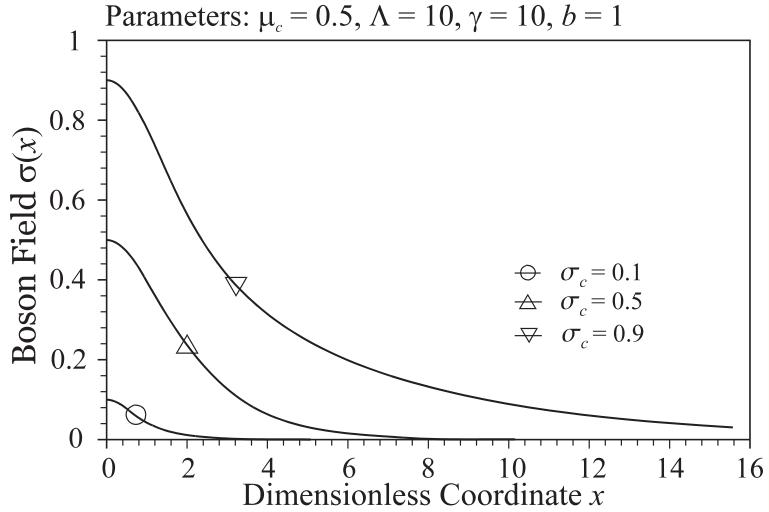


Figure 3: The bosonic density  $\sigma(x)$  in dependence on the parameter  $\sigma_c$ : “ $\circ$ ” -  $\sigma_c = 0.1$ ; “ $\triangle$ ” -  $\sigma_c = 0.5$ ; “ $\nabla$ ” -  $\sigma_c = 0.9$ .

From a physical point of view, the described behaviour of the dilaton field (and consequently the behaviour of the physical gravitational “constant”  $G_* A^2(\varphi)$ ) for the central values  $\sigma_c > \sigma_c^*$  seems to be strange. In order to clarify this situation, we have to take into account that in the range  $\sigma_c > \sigma_c^*$  (for the

fixed value of the central fermionic density  $\mu_c$ ) the star is unstable and, therefore, the mentioned range is not physically relevant. Such a behaviour has to be considered only as an interesting mathematical fact. In the domain of stability  $0 < \sigma_c < \sigma_c^*$ , as we have already seen, the dilaton field  $\varphi(x)$  has a normal physical behaviour - it decreases when the parameter  $\sigma_c$  increases.

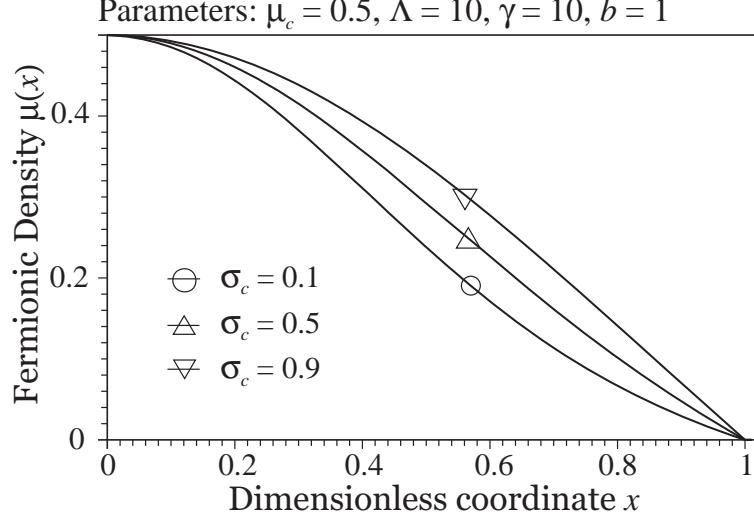


Figure 4: The fermionic density  $\mu(x)$  in dependence on the parameter  $\sigma_c$ : “ $\circ$ ” -  $\sigma_c = 0.1$ ; “ $\triangle$ ” -  $\sigma_c = 0.5$ ; “ $\nabla$ ” -  $\sigma_c = 0.9$ .

The dependence of the bosonic density  $\sigma(x)$  on the dimensionless coordinate  $x$  for three different values of  $\sigma_c$  is presented on Figure 3. The qualitative behaviour is the same for all three different values of  $\sigma_c$ . It approaches zero at infinity (rapidly when  $\sigma_c = 0.1$  and more slowly when  $\sigma_c$  increases).

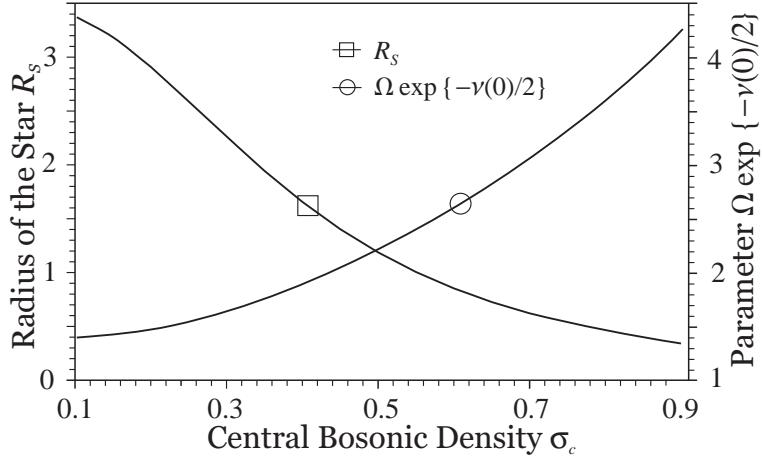


Figure 5: The radius of the star  $R_s$  and the quantity  $\Omega \exp(-\frac{\nu(0)}{2})$  as functions of the parameter  $\sigma_c$ :  $R_s$  - “ $\square$ ”;  $\Omega \exp(-\frac{\nu(0)}{2})$  - “ $\circ$ ”.

In the next Figure 4 the dependence of the fermionic density  $\mu(x)$  on the dimensionless coordinate  $x$  is presented for three different values of  $\sigma_c$ . The qualitative behaviour of the three curves is similar. In agreement with the initial assumption, it is nontrivial only within the star. It is seen that when the value of  $\sigma_c$  increases, the density  $\mu(x)$  increases as a whole, too. This fact is related to the effect of an increase of the gravitational field with the increase of  $\sigma_c$  - the star becomes more compact, which leads

to the greater density of matter, respectively to the function  $\mu(x)$ . The same may be seen in Figure 5 - when the central value  $\sigma_c$  increases, the radius of the star  $R_s$  decreases about 10 times.

From physical point of view it is important to get knowledge about the behaviour of the quantity  $\Omega \exp(-\frac{\nu(0)}{2})$  as a function of the central value  $\sigma_c$ . That quantity may be considered as the energy of one boson particle in the gravitational field yielded by the rest matter (in the Einstein frame). Figure 5 clearly shows that the quantity  $\Omega \exp(-\frac{\nu(0)}{2})$  increases along with  $\sigma_c$ . Such behaviour should be expected, because the energy of the system has to increase with the increase of the central density  $\sigma_c$  of the star.

## Concluding Remarks

Based on CANM an iteration method for solving the nonlinear BVP, describing a static spherically-symmetric boson-fermion star, is developed.

A linearization of the main equations of the star renders the original two-parametric nonlinear spectral problem to three two-point linear vector BVPs and a linear system of algebraic equations for the spectral parameters (the radius of the star  $R_s$  and the frequency  $\Omega$  of the bosonic field). A spline-collocation scheme of fourth order of approximation for solving numerically these BVPs is used.

Our basic physical result is that the structure and the properties of the star in the presence of a massive dilaton field depend essentially on both its fermionic and bosonic components. This shows that a careful investigation of these properties may give new real ways to discover physical effects of the hypothetical boson fields and dilaton field in stars.

## Appendix

For reader's convenience, we briefly explain the main ideas of CANM.

CANM can be treated as a particular case of the continuous analogues of iteration methods, strictly formulated and studied by M.K. Gavurin in 1958 (see the review in [41]). Among the number of papers devoted to the theoretical development and applications of CANM for solving wide classes of nonlinear equations, we will indicate the basic papers [29] - [33] as well as the reviews [34], [35].

Let us consider the nonlinear equation:

$$\chi(y) = 0, \quad (41)$$

where  $\chi(y)$  is an operator defined in a Banach space  $\mathbf{Y}$ . We suppose that the equation (41) has an isolated exact solution  $y^* \in \mathbf{Y}$ . Let the element  $y_0 \in \mathbf{Y}$  (an initial approximation to  $y^*$ ) be given. To solve equation (41), we can use an iteration process, usually taking it in the form:

$$y_{n+1} = y_n + \psi(y_n), \quad n = 0, 1, 2, \dots$$

Here,  $n$  indicates the number of iterations and  $\psi$  is an appropriate function, which carries  $\mathbf{Y}$  into itself and has the same zeroes as  $\chi$ .

The choice of the function  $\psi(y)$  depends on the kind of concrete iteration method used.

According to Gavurin's idea, for each iteration process of such kind it is possible to formulate the corresponding continuous analogue in the following way. Let us consider an abstract function  $y(t)$  of the independent continuous variable  $t \in [0, \infty)$  instead of the sequence  $y_0, y_1, \dots, y_n, \dots$ , and suppose that  $y(t_n) = y_n$  for each  $n$ . Then, we can introduce the derivative  $\dot{y}(t)$  instead of the increment  $y_{n+1} - y_n$  and replace (41) with the abstract initial value problem on the interval  $t \in [0, \infty)$

$$\dot{y}(t) = \psi(y), \quad y(0) = y_0. \quad (42)$$

Such a transition from a difference equation to a differential one has many advantages, both in pure theoretical and applied aspects.

In the case of Newton's method, we set  $\psi(y) = -\chi'(y)^{-1}\chi(y)$ , where  $\chi'(y)$  is the corresponding Fréchet derivative of  $\chi(y)$ . Then, the main equation of CANM, arising from (42), can be rewritten in the form:

$$\chi'(y)\dot{y} = -\chi(y). \quad (43)$$

Obviously, the above ODE has a significant first integral of the kind:

$$\chi(y(t)) = \chi(y_0) e^{-t}, \quad (44)$$

which means that  $\chi(y(t)) \rightarrow 0$  when  $t \rightarrow \infty$ .

Various theorems, based on (44), concerning the convergence of a path  $y(t)$  to the exact solution  $y^*$  have been proved. For example, a theorem [34], which guarantees the convergence of CANM for a simple BVP, is cited below.

The following BVP is considered:

$$-y'' + f(x, y) = 0, \quad x \in (0, 1), \quad (45)$$

$$y(0) = 0, \quad y(1) = 0. \quad (46)$$

**Theorem 1** *Let the BVP (45), (46) have an isolated solution  $y^*(x)$  and:*

- i) *the function  $f(x, y)$  have continuous partial derivatives up to the second order in some domain  $D$ ;*
- ii) *the linear BVP*

$$\begin{aligned} -w'' + f'_y(x, y)w &= 0, \quad x \in (0, 1), \\ w(0) &= 0, \quad w(1) = 0, \end{aligned}$$

*have only a trivial solution  $w(x) \equiv 0$  for every smooth function  $y(x) \in D$ ;*

- iii) *the initial approximation  $y_0(x) \in D$  be a smooth enough function satisfying:*

$$\| -y_0'' + f(x, y_0) \| \leq \varepsilon \quad \text{for } \varepsilon > 0.$$

*Then the system*

$$-w'' + f'_y(x, y)w = y'' - f(x, y), \quad \dot{y} = w,$$

*with BCs  $w(0, t) = 0$ ,  $w(1, t) = 0$ , and an initial condition  $y(x, 0) = y_0(x)$ , has in  $[0, 1] \cup [0, \infty)$  an unique solution, satisfying the relation:*

$$\lim_{t \rightarrow \infty} \|y(x, t) - y^*(x)\|_{C^2[0, 1]} = 0.$$

The numerical solution of CANM equation (43) is based on an appropriate scheme for discretization, which has to be stable for the asymptotic stability of the path  $y(t)$ . The most frequently used one is Euler's scheme (see the details in the above cited papers). At first, the linearized equation:

$$\chi'(y_n)w_n = -\chi(y_n), \quad (47)$$

is solved with respect to the increment  $w_n$ , and then the next approximation is obtained via the formula:

$$y_{n+1} = y_n + \tau_n w_n. \quad (48)$$

Here,  $0 < \tau_n \leq 1$  is an iteration parameter. When  $\tau_n \equiv 1$ , the classical Newton method is obtained. We note that the choice of  $\tau_n$  is important for the rapid convergence of the process. It is possible to choose this parameter so that the range of convergence is wider in comparison to the classical Newton's method [37], [33].

Theorems regarding the convergence of iterations (47), (48) for wide enough hypotheses as well as essential generalizations of CANM, are discussed in the above cited papers.

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